

ON THE NEGATIVE-ONE SHIFT FUNCTOR FOR FI-MODULES

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ABSTRACT. We show that the negative-one shift functor \tilde{S}_{-1} on the category of FI-modules is a left adjoint of the shift functor S and a right adjoint of the derivative functor D . We show that for any FI-module V , the coinduction QV of V is an extension of V by $\tilde{S}_{-1}V$.

1. INTRODUCTION

The shift functor S and the derivative functor D on the category of FI-modules have played an essential role in several recent works, for example, [1, 4, 5, 6, 7, 9]. The modest goal of this article is to explain how they are related to the negative-one shift functor \tilde{S}_{-1} introduced in [2]. We also explain how the coinduction functor Q defined in [3] is related to \tilde{S}_{-1} .

Let us begin by recalling some definitions.

Let \mathbb{k} be a commutative ring. Let \mathbf{C} be a small category. A \mathbf{C} -module is a functor from \mathbf{C} to the category of \mathbb{k} -modules. A *homomorphism of \mathbf{C} -modules* is a natural transformation of functors. For any \mathbf{C} -module V and $X \in \text{Ob}(\mathbf{C})$, we write V_X for $V(X)$.

Let FI be the category whose objects are the finite sets and whose morphisms are the injective maps.

Fix a one-element set $\{\star\}$ and define a functor $\sigma : \text{FI} \rightarrow \text{FI}$ by $X \mapsto X \sqcup \{\star\}$ for each finite set X . If $f : X \rightarrow Y$ is a morphism in FI , then $\sigma(f) : X \sqcup \{\star\} \rightarrow Y \sqcup \{\star\}$ is the map $f \sqcup \text{id}_{\{\star\}}$. Following [2, Definition 2.8], the *shift functor* S from the category of FI-modules to itself is defined by $SV = V \circ \sigma$ for every FI-module V .

Suppose V is an FI-module. For any finite set X , one has $(SV)_X = V_{X \sqcup \{\star\}}$. There is a natural FI-module homomorphism $\iota : V \rightarrow SV$ whose components $V_X \rightarrow (SV)_X$ are defined by the inclusion maps $X \hookrightarrow X \sqcup \{\star\}$. We denote by DV the cokernel of $\iota : V \rightarrow SV$. Following [1, Definition 3.2], we call the functor $D : V \mapsto DV$ the *derivative functor* on the category of FI-modules.

For any finite set X , let

$$(\tilde{S}_{-1}V)_X := \bigoplus_{x \in X} V_{X \setminus \{x\}}.$$

Suppose $f : X \rightarrow Y$ is an injective map between finite sets. For any $x \in X$, the map f restricts to an injective map $f|_{X \setminus \{x\}} : X \setminus \{x\} \rightarrow Y \setminus \{f(x)\}$. We let

$$f_* : (\tilde{S}_{-1}V)_X \longrightarrow (\tilde{S}_{-1}V)_Y$$

by the \mathbb{k} -linear map whose restriction to the direct summand $V_{X \setminus \{x\}}$ is the map

$$(f|_{X \setminus \{x\}})_* : V_{X \setminus \{x\}} \longrightarrow V_{Y \setminus \{f(x)\}}.$$

This defines an FI-module $\tilde{S}_{-1}V$. We call the functor $\tilde{S}_{-1} : V \mapsto \tilde{S}_{-1}V$ the *negative-one shift functor* on the category of FI-modules. (In [2, Definition 2.19], a negative shift functor \tilde{S}_{-a} is defined for each integer $a \geq 1$.)

After recalling some generalities in the next section, we shall show that the negative-one shift functor \tilde{S}_{-1} is a left adjoint functor to the shift functor S and a right adjoint functor to the derivative functor D ; we shall also show that for any FI-module V , the coinduction QV of V is an extension of V by $\tilde{S}_{-1}V$. These properties seem to have gone unnoticed; for instance, the paper [6] constructed a left adjoint functor of S using the category algebra of FI and called it the induction functor.

Notations. Suppose \mathbf{C} is a small category. For any $X, Y \in \text{Ob}(\mathbf{C})$, we write $\mathbf{C}(X, Y)$ for the set of morphisms from X to Y .

We write $\mathbf{C}\text{-Mod}$ for the category of \mathbf{C} -modules. Suppose $V, W \in \mathbf{C}\text{-Mod}$. We write $\text{Hom}_{\mathbf{C}\text{-Mod}}(V, W)$ for the \mathbb{k} -module of all \mathbf{C} -module homomorphisms from V to W . If $\phi \in \text{Hom}_{\mathbf{C}\text{-Mod}}(V, W)$, we write $\phi_X : V_X \rightarrow W_X$ for the component of ϕ at $X \in \text{Ob}(\mathbf{C})$.

2. GENERALITIES ON ADJOINT FUNCTORS

Let \mathbf{C} and \mathbf{C}' be small categories, and $F : \mathbf{C}\text{-Mod} \rightarrow \mathbf{C}'\text{-Mod}$ a functor.

For any $X \in \text{Ob}(\mathbf{C})$, we define the \mathbf{C} -module $M(X)$ by

$$M(X)_Y := \mathbb{k}\mathbf{C}(X, Y) \quad \text{for each } Y \in \text{Ob}(\mathbf{C}),$$

that is, $M(X)$ is the composition of the functor $\mathbf{C}(X, -)$ followed by the free \mathbb{k} -module functor.

For any morphism $f \in \mathbf{C}(X, Y)$, we have a \mathbf{C} -module homomorphism

$$\rho_f : M(Y) \longrightarrow M(X), \quad g \mapsto gf,$$

where $g \in \mathbf{C}(Y, Z)$ for any $Z \in \text{Ob}(\mathbf{C})$. Hence, we obtain a \mathbf{C}' -module homomorphism

$$F(\rho_f) : F(M(Y)) \longrightarrow F(M(X)).$$

Definition 1. Define a functor $F^\dagger : \mathbf{C}'\text{-Mod} \rightarrow \mathbf{C}\text{-Mod}$ by

$$(F^\dagger(V))_X := \text{Hom}_{\mathbf{C}'\text{-Mod}}(F(M(X)), V) \quad \text{for each } V \in \mathbf{C}'\text{-Mod}, X \in \text{Ob}(\mathbf{C}).$$

For any morphism $f \in \mathbf{C}(X, Y)$, the map $f_* : (F^\dagger(V))_X \rightarrow (F^\dagger(V))_Y$ is defined by

$$f_*(\phi) := \phi \circ F(\rho_f) \quad \text{for each } \phi \in (F^\dagger(V))_X.$$

Proposition 2. *Let \mathbf{C} and \mathbf{C}' be small categories. Let $F : \mathbf{C}\text{-Mod} \rightarrow \mathbf{C}'\text{-Mod}$ be a right exact functor which transforms direct sums to direct sums. Then F^\dagger is a right adjoint functor of F , that is, there is a natural isomorphism*

$$\text{Hom}_{\mathbf{C}'\text{-Mod}}(F(V), W) \cong \text{Hom}_{\mathbf{C}\text{-Mod}}(V, F^\dagger(W)),$$

where $V \in \mathbf{C}\text{-Mod}$ and $W \in \mathbf{C}'\text{-Mod}$.

Proof. This is a well-known result for modules over rings with identity, see [10, Theorem 5.51]. For a proof in our setting, see [8, Proposition 1.3 and Theorem 2.1]. \square

The proofs of the main results of this article are essentially exercises in applications of Proposition 2; nevertheless, they do not appear to be obvious.

Remark 3. In all our applications of Proposition 2, we have a pair of functors $F, G : \mathbf{C}\text{-Mod} \rightarrow \mathbf{C}\text{-Mod}$ with the following properties. For each $X \in \mathbf{Ob}(\mathbf{C})$, there exists $X_i \in \mathbf{Ob}(\mathbf{C})$ for $i \in I_X$ (where I_X is a finite indexing set) such that there is a \mathbf{C} -module isomorphism

$$\eta_X : \bigoplus_{i \in I_X} M(X_i) \longrightarrow F(M(X)).$$

Moreover, for each $V \in \mathbf{C}\text{-Mod}$, there is a decomposition $G(V)_X = \bigoplus_{i \in I_X} V_{X_i}$ such that for any $\psi \in \mathbf{Hom}_{\mathbf{C}\text{-Mod}}(V, W)$, one has $G(\psi)_X = \bigoplus_{i \in I_X} \psi_{X_i}$. It follows then that there is a \mathbb{k} -module isomorphism

$$\alpha_X : (F^\dagger(V))_X \longrightarrow G(V)_X, \quad \phi \mapsto \sum_{i \in I_X} \phi_{X_i}((\eta_X)_{X_i}(\text{id}_{X_i})),$$

where $\text{id}_{X_i} \in M(X_i)_{X_i}$. Therefore, if F is right exact and sends direct sums to direct sums, then by Proposition 2, to show that G is a right adjoint functor of F , it suffices to verify that the collection of \mathbb{k} -module isomorphisms α_X for $X \in \mathbf{Ob}(\mathbf{C})$ are compatible with the \mathbf{C} -module structures on $F^\dagger(V)$ and $G(V)$.

3. LEFT ADJOINT OF THE SHIFT FUNCTOR

Let X be a finite set. By [2, Lemma 2.17], there is an isomorphism

$$\eta_X : M(X \sqcup \{\star\}) \longrightarrow \tilde{S}_{-1}M(X) \tag{1}$$

defined as follows. For any finite set Y and injective map $f : X \sqcup \{\star\} \rightarrow Y$, one has $f|_X : X \rightarrow Y \setminus \{f(\star)\}$. Let $\eta_X(f)$ be the element $f|_X$ of the direct summand $M(X)_{Y \setminus \{f(\star)\}}$ of $(\tilde{S}_{-1}M(X))_Y$.

Theorem 4. *The functor \tilde{S}_{-1} is a left adjoint of the shift functor $S : \mathbf{FI}\text{-Mod} \rightarrow \mathbf{FI}\text{-Mod}$.*

Proof. The functor \tilde{S}_{-1} is exact and transforms direct sums to direct sums. By Proposition 2, we need to show that the functors \tilde{S}_{-1}^\dagger and S are isomorphic.

Let V be an FI-module. Let X be any finite set. From Definition 1 and (1), we have a \mathbb{k} -module isomorphism

$$\alpha_X : (\tilde{S}_{-1}^\dagger V)_X \longrightarrow (SV)_X, \quad \phi \mapsto \phi_{X \sqcup \{\star\}}((\eta_X)_{X \sqcup \{\star\}}(\text{id}_{X \sqcup \{\star\}})).$$

We claim that this collection of \mathbb{k} -module isomorphisms over all finite sets X are compatible with the FI-module structures on $\tilde{S}_{-1}^\dagger V$ and SV .

To verify the claim, let $f : X \rightarrow Y$ be an injective map between finite sets, and let $\phi \in (\tilde{S}_{-1}^\dagger V)_X$. Then one has:

$$\begin{aligned} \alpha_Y(f_*(\phi)) &= f_*(\phi)_{Y \sqcup \{\star\}}((\eta_Y)_{Y \sqcup \{\star\}}(\text{id}_{Y \sqcup \{\star\}})) = \phi_{Y \sqcup \{\star\}}(\tilde{S}_{-1}(\rho_f)_{Y \sqcup \{\star\}}((\eta_Y)_{Y \sqcup \{\star\}}(\text{id}_{Y \sqcup \{\star\}}))) \\ &= \phi_{Y \sqcup \{\star\}}((\eta_X)_{Y \sqcup \{\star\}}(f \sqcup \text{id}_{\{\star\}})) = \phi_{Y \sqcup \{\star\}}((f \sqcup \text{id}_{\{\star\}})_*((\eta_X)_{X \sqcup \{\star\}}(\text{id}_{X \sqcup \{\star\}}))) \\ &= (f \sqcup \text{id}_{\{\star\}})_*(\phi_{X \sqcup \{\star\}}((\eta_X)_{X \sqcup \{\star\}}(\text{id}_{X \sqcup \{\star\}}))) = f_*(\alpha_X(\phi)). \end{aligned}$$

□

4. RIGHT ADJOINT OF THE DERIVATIVE FUNCTOR

Let X be a finite set. It is known that there is an isomorphism

$$\Theta_X : M(X) \oplus \left(\bigoplus_{x \in X} M(X \setminus \{x\}) \right) \longrightarrow SM(X), \quad (2)$$

see [2, Proof of Proposition 2.12]. We need this isomorphism explicitly. The restriction of Θ to the direct summand $M(X)$ is $\iota : M(X) \rightarrow SM(X)$. To define the restriction of Θ to the direct summand $M(X \setminus \{x\})$, we shall use the following notation.

Notation 5. Suppose $f : X \rightarrow Y$ is a map between finite sets. Suppose $\{w\}$ and $\{z\}$ are any one-element sets. We write $f \sqcup (w \rightarrow z)$ for the map $X \sqcup \{w\} \rightarrow Y \sqcup \{z\}$ whose restriction to X is f and which sends w to z .

Let Y be a finite set. The map $\Theta_X : M(X \setminus \{x\})_Y \rightarrow SM(X)_Y$ is defined by

$$f \mapsto f \sqcup (x \rightarrow \star)$$

for each injective map $f : X \setminus \{x\} \rightarrow Y$.

Theorem 6. *The functor \tilde{S}_{-1} is a right adjoint of the derivative functor $D : \text{FI-Mod} \rightarrow \text{FI-Mod}$.*

Proof. The functor D is right exact and transforms direct sums to direct sums. By Proposition 2, we need to show that the functors D^\dagger and \tilde{S}_{-1} are isomorphic.

Let X be a finite set. It follows from above that we have an isomorphism

$$\theta_X : \bigoplus_{x \in X} M(X \setminus \{x\}) \longrightarrow DM(X), \quad (3)$$

where θ_X is defined by the composition of Θ_X with the quotient map $\pi_X : SM(X) \rightarrow DM(X)$.

Let V be an FI-module. From Definition 1 and (3), we have a \mathbb{k} -module isomorphism

$$\beta_X : (D^\dagger V)_X \longrightarrow (\tilde{S}_{-1} V)_X, \quad \phi \mapsto \sum_{x \in X} \phi_{X \setminus \{x\}}((\theta_X)_{X \setminus \{x\}}(\text{id}_{X \setminus \{x\}})).$$

We claim that this collection of \mathbb{k} -module isomorphisms over all finite sets X are compatible with the FI-module structures on $D^\dagger V$ and $\tilde{S}_{-1} V$.

To verify the claim, let $f : X \rightarrow Y$ be an injective map between finite sets, and let $\phi \in (D^\dagger V)_X$. Then one has:

$$\begin{aligned} \beta_Y(f_*(\phi)) &= \sum_{y \in Y} f_*(\phi)_{Y \setminus \{y\}}((\theta_Y)_{Y \setminus \{y\}}(\text{id}_{Y \setminus \{y\}})) = \sum_{y \in Y} \phi_{Y \setminus \{y\}}(D(\rho_f)_{Y \setminus \{y\}}((\theta_Y)_{Y \setminus \{y\}}(\text{id}_{Y \setminus \{y\}}))) \\ &= \sum_{y \in Y} \phi_{Y \setminus \{y\}}(\pi_X((\text{id}_{Y \setminus \{y\}} \sqcup (y \rightarrow \star)) \circ f)) = \sum_{x \in X} \phi_{Y \setminus \{f(x)\}}((\theta_X)_{Y \setminus \{f(x)\}}(f|_{X \setminus \{x\}})) \\ &= \sum_{x \in X} \phi_{Y \setminus \{f(x)\}}((f|_{X \setminus \{x\}})_*((\theta_X)_{X \setminus \{x\}}(\text{id}_{X \setminus \{x\}}))) \\ &= \sum_{x \in X} (f|_{X \setminus \{x\}})_*(\phi_{X \setminus \{x\}}((\theta_X)_{X \setminus \{x\}}(\text{id}_{X \setminus \{x\}}))) = f_*(\beta_X(\phi)). \end{aligned}$$

□

5. COINDUCTION FUNCTOR

The coinduction functor Q on the category of FI-modules was defined in [3, Definition 4.1] as S^\dagger . It is a right adjoint functor of S by Proposition 2 (see [3, Lemma 4.2]). In this section, we give an explicit description of Q in terms of \tilde{S}_{-1} .

Notation 7. Suppose $f : X \rightarrow Y$ is a map between finite sets. If $y \in Y \setminus f(X)$, then define

$$\partial_y f : X \rightarrow Y \setminus \{y\}$$

by $\partial_y f(x) := f(x)$ for each $x \in X$.

Notation 8. Suppose V is an FI-module. If $f : X \rightarrow Y$ is a morphism in FI, define the map

$$\partial f_* : V_X \longrightarrow (\tilde{S}_{-1}V)_Y, \quad v \mapsto \sum_{y \in Y \setminus f(X)} (\partial_y f)_*(v),$$

where $(\partial_y f)_* : V_X \rightarrow V_{Y \setminus \{y\}}$ is defined by the FI-module structure of V .

It is easily checked that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are morphisms in FI, then one has:

$$\partial(gf)_* = (\partial g_*)f_* + g_*(\partial f_*). \quad (4)$$

We define an FI-module $Q'V$ as follows. For each finite set X , let

$$(Q'V)_X := V_X \oplus (\tilde{S}_{-1}V)_X. \quad (5)$$

If $f : X \rightarrow Y$ is a morphism in FI, then define $f_* : (Q'V)_X \rightarrow (Q'V)_Y$ to be

$$\begin{pmatrix} f_* & 0 \\ \partial f_* & f_* \end{pmatrix},$$

where we use column notation for the direct sum in (5). It follows from (4) that $Q'V$ is an FI-module.

Theorem 9. *The coinduction functor $Q : \text{FI-Mod} \rightarrow \text{FI-Mod}$ is isomorphic to the functor $Q' : V \mapsto Q'V$.*

Proof. Let V be an FI-module. Let X be any finite set. One has $QV = S^\dagger V$ by definition of Q . From Definition 1 and (2), we have a \mathbb{k} -module isomorphism

$$\gamma_X : (S^\dagger V)_X \longrightarrow (Q'V)_X, \quad \phi \mapsto \phi_X((\Theta_X)_X(\text{id}_X)) + \sum_{x \in X} \phi_{X \setminus \{x\}}((\Theta_X)_{X \setminus \{x\}}(\text{id}_{X \setminus \{x\}})).$$

We claim that this collection of \mathbb{k} -module isomorphisms over all finite sets X are compatible with the FI-module structures on $S^\dagger V$ and $Q'V$.

To verify the claim, let $f : X \rightarrow Y$ be an injective map between finite sets, and let $\phi \in (S^\dagger V)_X$. Then one has:

$$\begin{aligned}
\gamma_Y(f_*(\phi)) &= f_*(\phi)_Y((\Theta_Y)_Y(\text{id}_Y)) + \sum_{y \in Y} f_*(\phi)_{Y \setminus \{y\}}((\Theta_Y)_{Y \setminus \{y\}}(\text{id}_{Y \setminus \{y\}})) \\
&= \phi_Y(S(\rho_f)_Y((\Theta_Y)_Y(\text{id}_Y))) + \sum_{y \in Y} \phi_{Y \setminus \{y\}}(S(\rho_f)_{Y \setminus \{y\}}((\Theta_Y)_{Y \setminus \{y\}}(\text{id}_{Y \setminus \{y\}}))) \\
&= \phi_Y((\Theta_X)_Y(f)) + \sum_{y \in Y} \phi_{Y \setminus \{y\}}((\text{id}_{Y \setminus \{y\}} \sqcup (y \rightarrow \star)) \circ f) \\
&= \phi_Y((\Theta_X)_Y(f)) + \sum_{y \in Y \setminus f(X)} \phi_{Y \setminus \{y\}}((\Theta_X)_{Y \setminus \{y\}}(\partial_y f)) \\
&\quad + \sum_{x \in X} \phi_{Y \setminus \{f(x)\}}((\Theta_X)_{Y \setminus \{f(x)\}}(f|_{X \setminus \{x\}})) \\
&= \phi_Y(f_*((\Theta_X)_X(\text{id}_X))) + \sum_{y \in Y \setminus f(X)} \phi_{Y \setminus \{y\}}((\partial_y f)_*((\Theta_X)_X(\text{id}_X))) \\
&\quad + \sum_{x \in X} \phi_{Y \setminus \{f(x)\}}((f|_{X \setminus \{x\}})_*((\Theta_X)_{X \setminus \{x\}}(\text{id}_{X \setminus \{x\}}))) \\
&= f_*(\phi_X((\Theta_X)_X(\text{id}_X))) + \sum_{y \in Y \setminus f(X)} (\partial_y f)_*(\phi_X((\Theta_X)_X(\text{id}_X))) \\
&\quad + \sum_{x \in X} (f|_{X \setminus \{x\}})_*(\phi_{X \setminus \{x\}}((\Theta_X)_{X \setminus \{x\}}(\text{id}_{X \setminus \{x\}}))) = f_*(\gamma_X(\phi)).
\end{aligned}$$

□

Let FB be the category whose objects are the finite sets and whose morphisms are the bijections. Then FB is a subcategory of FI and so there is a natural forgetful functor from FI-Mod to FB-Mod (see [1]).

Corollary 10. *Let V be an FI-module. Then there is a short exact sequence*

$$0 \longrightarrow \tilde{S}_{-1}V \longrightarrow QV \longrightarrow V \longrightarrow 0$$

of FI-modules which splits after applying the forgetful functor to the category of FB-modules.

Proof. Immediate from Theorem 9 and the definition of Q' . □

From Corollary 10 and (1), we recover [3, Theorem 1.3] for FI.

Remark 11. In [3], for any finite field \mathbb{F}_q , Gan and Li also studied the coinduction functor for VI-modules where VI is the category whose objects are the finite dimensional vector spaces over \mathbb{F}_q and whose morphisms are the injective linear maps. Similarly to FI-modules, the coinduction functor for VI-modules is defined as S^\dagger where S is the shift functor for VI-modules. However, we do not know of analogues of the negative-one shift functor \tilde{S}_{-1} and the derivative functor D for VI-modules.

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